

An Introduction to the
TRU Math Dimensions
Teaching for Robust Understanding in Mathematics

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An Introduction to the TRU Math Dimensions

Teaching for Robust Understanding in Mathematics

Overview

This document provides the theoretical rationale underlying the Teaching for Robust Understanding in Mathematics (TRU Math) analytic scheme. The TRU Math scheme consists of an analytic framework for characterizing important dimensions of mathematics classroom activity and a scoring rubric for capturing their presence in instruction. The scheme has two parts: a general frame that applies to all mathematical classrooms, and a content-specific component that applies to solving contextual algebraic problems. The general part of TRU Math delineates a measurement scheme that focuses on five minimally overlapping dimensions of mathematics classroom activity. Each of these five dimensions captures an essential aspect of productive mathematics classrooms – classrooms that produce powerful mathematical thinkers.

The Five Dimensions of Mathematically Powerful Classrooms:

The Mathematics	Cognitive Demand	Access to Mathematical Content	Agency, Authority, and Identity	Uses of Assessment
<i>The extent to which the mathematics discussed is focused and coherent, and to which connections between procedures, concepts and contexts (where appropriate) are addressed and explained. Students should have opportunities to learn important mathematical content and practices, and to develop productive mathematical habits of mind.</i>	<i>The extent to which classroom interactions create and maintain an environment of productive intellectual challenge conducive to students' mathematical development. There is a happy medium between spoon-feeding mathematics in bite-sized pieces and having the challenges so large that students are lost at sea.</i>	<i>The extent to which classroom activity structures invite and support the active engagement of all of the students in the classroom with the core mathematics being addressed by the class. No matter how rich the mathematics being discussed, a classroom in which a small number of students get most of the "air time" is not equitable.</i>	<i>The extent to which students have opportunities to conjecture, explain, make mathematical arguments, and build on one another's ideas, in ways that contribute to their development of agency (the capacity and willingness to engage mathematically) and authority (recognition for being mathematically solid), resulting in positive identities as doers of mathematics.</i>	<i>The extent to which the teacher solicits student thinking and subsequent instruction responds to those ideas, by building on productive beginnings or addressing emerging misunderstandings. Powerful instruction "meets students where they are" and gives them opportunities to move forward.</i>

We conjecture that the general portion of the TRU Math scheme spans the major dimensions of productive mathematics classrooms – that is, at this level of analytical grain size, no further dimensions will be necessary for analysis¹. We are empirically exploring the conjecture that classrooms that consistently earn high scores on TRU Math will produce students who do well on tests of mathematical understanding, thinking, and problem solving.

The goal of this document is to introduce both parts of the TRU math framework and to provide support and elaboration from relevant literature for each for each of the dimensions. This document does not provide sufficient detail to be used to score classroom observations. That level of specificity is provided in the TRU Math scoring rubric and scoring guide.

The dimensions of TRU Math can serve as foci for professional development in mathematics instruction: Helping teachers become increasingly skilled at creating learning environments that do well along the dimensions represented in TRU Math should help their students become more powerful mathematical thinkers and problem solvers. The *TRU Math Conversation guides*, available at <http://ats.berkeley.edu/tools.html> and <http://map.mathshell.org/materials/index.php>, are tools for supporting reflection along on the key dimensions of the TRU Math framework.

TRU Math was developed partially under the aegis of the Algebra Teaching Study, which has a focus on supporting students’ ability to grapple successfully with Contextual Algebraic Tasks – “real world” problems, typically described in words, whose solutions typically employ algebraic methods and representations. Thus, while the five dimensions identified above are assumed to always be in play in any mathematics classroom, TRU Math also includes an algebra-specific content elaboration, which offers the following definition and criteria:

Content Elaboration for Contextual Algebraic Tasks

The extent to which students are supported in dealing with complex modeling and applications problems, which typically call for understanding complex problem contexts (most frequently described in text), identifying relevant variables and the relationships between them, representing those variables and relationships symbolically, operating on the symbols, and interpreting the results. Instruction supports students in:

1. reading and interpreting text, and understanding the contexts described in problem statements

2. identifying salient quantities in a problem and the relationships between these quantities

3. generating and using appropriate algebraic representations

4. executing calculations and procedures with precision, and checking plausibility of results

5. providing convincing explanations that give further insight into the depth of students’ algebraic thinking.

¹ Of course, other possible decompositions might highlight different things (e.g., one might start with equity as a dimension). Moreover, as we make clear below, the scheme itself as a measurement tool is of necessity selective; it is impossible to be comprehensive. The claim here is that the five dimensions, broadly construed, encompass the essentials of productive mathematics classrooms.

Our expectation is that similar domain-specific foci would be useful for versions of TRU Math in other content domains as well. (For example, one might develop rubrics for student understanding of the proof process when focusing on a robust understanding of geometry.)

Before turning to the detailed elaboration of the dimensions in TRU Math, we turn to some issues concerning its nature and organization, and we situate this document within the family of documents describing our analytic work.

The concept of minimal overlap in the TRU Math dimensions

When one attempts to focus on the central dimensions of any complex activity or system, there is necessarily some overlap between dimensions. For example, a classic decomposition for purposes of anatomical study is to consider the circulatory system, skeletal system, muscular system, respiratory system, nervous system, and so on. One can speak of each system as being somewhat independent – e.g., one can analyze the circulatory and respiratory systems on their own terms – while recognizing their inter-dependence (if the circulatory system is compromised, for example, the respiratory system will not be able to function very well). This is the ideal kind of decomposition for dimensions in an analytic scheme. We have tried to frame the five dimensions in TRU Math so that their overlap is as small as possible. Non-overlap is impossible: for example, the mathematical richness of the tasks that students are asked to work (dimension 1 of the scheme) will shape opportunities for cognitive demand (dimension 2) and for developing an identity as a doer of mathematics (dimension 4). Wherever possible, however, we have worded the rubrics for different dimensions to minimize overlap.

The issue of equity and its placement in TRU Math

One critically important feature of classroom activities is the extent to which *all* students are being helped to become powerful mathematical thinkers and problem solvers (see, e.g., Moses, 2001; Schoenfeld, 2002). This issue of equity arises in all aspects of instruction, and thus in all dimensions of the scheme. That is, cognitive demand may be roughly equal for all students, or it may be that the teacher reduces cognitive demand for some students, while maintaining it for others. Similarly, for each of the six dimensions, students may vary in what they experience in the classroom, or they may have roughly equivalent experiences.

It is beyond the scope of this scheme to capture the variability of student experience for each of the six dimensions.

The one dimension in which variation in student experience is captured is Dimension 3 – Access to Mathematical Content. Scoring on this dimension reflects whether all, or only some, students have access to potentially rich classroom activities (e.g., when the class is divided into different “ability groups,” the teacher routinely calls on a subset of high-performing students, or some students are excluded or given menial tasks when the class engages in small group work).

Where TRU Math fits in the context of Algebra Teaching Study and Mathematics Assessment Project Work

TRU Math was developed under the aegis of The Algebra Teaching Study (ATS)², NSF Grant DRL-0909815, and The Mathematics Assessment Project (MAP)³, Bill and Melinda Gates Foundation Grant OPP53342) and it is related to work in each. The major product of the MAP work is a collection of 100 *Formative Assessment Lessons* (FALs), which are designed to support teachers in crafting classroom environments that embody powerful, student-centered instruction. TRU Math will be used as a measure of those classroom environments and, potentially of teacher growth as teachers become more fluent in the use of the FALs. The explicit rationale for ATS was the exploration and documentation of the relationship between powerful learning environments in algebra and students' performance on contextual algebraic tasks. This document is part of a collection of documents presenting and defining the scheme, its validation, and its use. Those include:

- Schoenfeld (2013), which describes the rationale for and evolution of TRU Math
- The Introduction to the TRU Math Document Suite
- The TRU Math Rubric, Release Version Alpha
- This Introduction to the TRU Math Dimensions, Release Version Alpha
- The TRU Math Scoring Guide
- Conference papers describing project research
- The TRU Math Conversation Guides, Release Version Alpha
- Statistical data relating classroom scores on TRU Math with student scores on MARS tests.
- A paper comparing the affordances and constraints of a range of classroom analysis tools.

These documents are or will be posted on the ATS web site < <http://ats.berkeley.edu/>> and the MAP web site <<http://map.mathshell.org/materials/index.php>>.

The balance of this document elaborates on the rationale for each dimension, and briefly describes the kinds of classroom activities that will result in scores of 1, 2, or 3 on the TRU Math rubric. The rubric has separate sub-rubrics for different classroom activity structures (whole class, small group work, student presentations, and individual seat work). Scores are assigned for each of these activity structures, and aggregated across a full lesson observation. Detailed instructions for assigning scores are given in the TRU Math Scoring Guide.

² The ATS web site can be accessed at < <http://ats.berkeley.edu/>>

³ The MAP web site can be accessed at <<http://map.mathshell.org/materials/index.php>>.

Dimension 1: The Mathematics

The extent to which the mathematics discussed is focused and coherent, and to which connections between procedures, concepts and contexts (where appropriate) are addressed and explained. Students should have opportunities to learn important mathematical content and practices, and to develop productive mathematical habits of mind.

Background

A major finding regarding instructional materials in the U.S., documented by the TIMSS studies (see, e.g., <http://www2.ed.gov/rschstat/research/progs/mathscience/schmidt.html>), is that curricular materials in the U.S. have tended to be “a mile wide and an inch deep,” covering many topics in a superficial way rather than concentrating attention on a smaller number of big ideas and developing those ideas carefully. Cross-national studies suggest that mathematics experiences in school should be focused and coherent in order to improve mathematics achievement (see, e.g., Ginsburg, Leinwand, & Decker, 2009). Recent curricular documents have placed significant emphasis on focus and coherence, including emphases on understanding, reasoning, and justification as the glue that give mathematics its coherence. This is clearly argued in the Common Core State Standards in Mathematics:

Toward greater focus and coherence: “For over a decade, research studies of mathematics education in high-performing countries have pointed to the conclusion that the mathematics curriculum in the United States must become substantially more focused and coherent in order to improve mathematics achievement in this country. To deliver on the promise of common standards, the standards must address the problem of a curriculum that is ‘a mile wide and an inch deep.’ These Standards are a substantial answer to that challenge” (Common Core State Standards Initiative, 2010, p. 3).

Understanding mathematics: “These Standards define what students should understand and be able to do in their study of mathematics. Asking a student to understand something means asking a teacher to assess whether the student has understood it. But what does mathematical understanding look like? One hallmark of mathematical understanding is the ability to justify, in a way appropriate to the student’s mathematical maturity, why a particular mathematical statement is true or where a mathematical rule comes from.” (Common Core State Standards Initiative, 2010, p. 3)

The first dimension of TRU Math, *The Mathematics*, focuses on the question of whether students experience mathematics as a set of isolated facts, procedures and concepts, to be rehearsed, memorized, and applied; or whether they experience mathematics as a coherent discipline, in which symbolization is a meaningful process and procedures can be re-derived if need be. Rich mathematics instruction provides students with opportunities to engage with centrally important mathematical content (key concepts and skills) and practices (CCSSM, 2010; NCTM, 1989, 2000; NRC, 2001). Mathematics instruction that is coherent, connected, and focused, and that is oriented toward sense making, helps students to develop networks of understandings that are robust and regenerative, and helps them to develop mathematical habits of mind that help them to make effective use of mathematics.

Rich mathematics instruction provides students with the opportunity to engage in, and develop, the mathematical practices highlighted in the CCSSM. In particular, students should have opportunities to engage and persevere in problem solving; to produce and critique extended chains of reasoning; and to use mathematical concepts and representations to make sense of real world phenomena.

Mathematical focus and coherence in the mathematics classroom

Virtually all of the mathematics studied in K-12 and beyond can be experienced as the codification of patterns that cohere in some natural way – that is, as something that makes sense, rather than something arbitrary. Perhaps the best-known example of such sense making in elementary mathematics comes from Deborah Ball’s third grade class (Ball & Bass, 2003; Stylianides, 2007), in which her students were discussing even and odd numbers. Some students noticed that whenever they had added two odd numbers, the sum was even. They wondered if this would always be the case. Some students objected that, since the odd numbers “go on forever,” it would be impossible to test them all. One student went to the board to show why a specific case, seven plus nine, had to be even. As was their convention, she represented seven as three pairs, plus one left over. (The left-over unit was what made seven odd.) Then she represented nine as four pairs, plus one left over. When you add seven and nine, she said, the result was all of the pairs that you had before, plus the two left-overs. The two left-overs make another pair, so seven and nine together make a bunch of pairs. It *had to* be even. The class discussed this, and it came to realize that that seven plus nine was just an example. It didn’t make a difference what the first odd number was – it would be a bunch of pairs, with one left over. And it didn’t make a difference what the second odd number was – it would also be a bunch of pairs, with one left over. When you add them, you would get all the pairs you had before, and the two left-overs would make one more pair. So, it doesn’t matter what the two odd numbers were, their sum would be even.

Sense-making can and should start early, and continue throughout students’ experience with mathematics (Schoenfeld, 1990, 1992, 2012). Consider algebraic representations, and operations on them. Solving a pair of simultaneous equations by adding and subtracting, or by substitution, can either be seen as a set of almost-arbitrary manipulations, or as a set of procedures that are guided by the wish to isolate one variable so that its value can be determined. Different forms of the equation for a line in the Cartesian plane – e.g.,

$$y = mx + b,$$

$$Ax + By = C,$$

$$\frac{x}{a} + \frac{y}{b} = 1 \text{ and}$$

$$\frac{y-y_0}{x-x_0} = m,$$

can be (and often are) seen as separate equations, each with its own properties, and each to be memorized; or they can be seen as different representations of the same linear relationship, where any two pieces of information about the line (e.g., two points that lie on it; a point on it and the line’s slope; the slope and y-intercept; the two intercepts) determine the position of the line, and that any of the equations for a line can be transformed into any of the others. Typical classes of word problems (e.g., distance-rate-time problems, or mixture problems, or age problems), can be approached as

separate categories of problems, each of which has a particular algorithm for its solution; or they can be seen as representing various relationships that can be captured symbolically through sensible representational practices. In geometry, proof can be approached mechanically (with rules for filling in a two column chart) or as a way of making mathematically compelling arguments.

Research indicates that that when mathematics is taught as a set of isolated procedures to be memorized and applied by rote, the result is fragile knowledge (slight changes in problem context stymie the students) and the development of a series of counterproductive beliefs and practices – for example, that if one has forgotten the formula for dealing with a situation, one is simply stuck. This stands in stark contrast to the beliefs and practices developed by students who have experienced mathematics as something that should make sense, and whose formulas and procedures can be re-generated when needed (Boaler, 2002; Chazan, 2000; Lampert, 1990b, 2001; Schoenfeld, 1988, 1992).

Caveats and Connections to Other Dimensions

Specifics for contextually rich algebra tasks are delineated below, in the section “Content Elaboration: for Contextual Algebraic Tasks”⁴. The rubrics in Dimension 1 can to be applied in *any* mathematics classrooms; the rubrics in the content elaboration add depth when they are used in algebra classrooms (which is where we developed the scheme).

How TRU Math captures Mathematical focus and coherence

Broadly speaking, *the mathematics* will be coded as 1, 2, or 3 in TRU Math as follows:

The Mathematics	
1	Classroom activities are unfocused or skills-oriented, lacking opportunities to engage in key practices such as reasoning and problem solving.
2	Activities are primarily skills-oriented, with cursory connections between procedures, concepts and contexts (where appropriate) and minimal attention to key practices.
3	Classroom activities support meaningful connections between procedures, concepts and contexts (where appropriate) and provide opportunities to engage in key practices.

The overarching question is, to what extent does the classroom environment provide students with the opportunity to engage in mathematical sense making and to develop the kinds of connected understandings discussed in this section? If a classroom episode has a skills-oriented focus, attention is not given to the conceptual underpinnings of mathematical procedures and there is no opportunity to develop and employ mathematical practices, then that episode of instruction will be assigned a score of 1 for mathematical focus and coherence. (The TIMSS U.S. geometry video is an archetype of this kind of instruction. In that video students are asked short factual questions only; they do not have the opportunity to engage or persevere in problem solving, or to make mathematical arguments.) If some

⁴ As noted, analogous detail can be developed for analyses of classrooms devoted to other topics.

attention is given to concepts and connections between concepts and procedures, or the students have some opportunity to engage in the practices described in various standards documents (CCSSM, 2010; NCTM, 1989, 2000; NRC, 2001), the episode will be assigned a score of 2. Scores of 3 will mean that focused attention is given to grounding procedures in their conceptual underpinnings and that significant attention is given to mathematical sense making – students are asked to reason and make connections.

Dimension 2: Cognitive Demand

The extent to which classroom interactions create and maintain an environment of productive intellectual challenge that is conducive to students' mathematical development. There is a happy medium between spoon-feeding mathematics in bite-sized pieces and having the challenges so large that students are lost at sea.

Background

In a series of articles, Stein, Henningsen, and colleagues (Henningsen & Stein, 1997; Stein, Engle, Smith, & Hughes, 2008; Stein, Grover, & Henningsen, 1996) explored the role of classroom discourse in either maintaining or diluting the potential mathematical richness of tasks with which students engage. Henningsen and Stein (1997) document five factors that appear to be “prime influences associated with maintaining student engagement at the level of doing mathematics.” Dimension 2 of TRU Math, *Cognitive Demand*, focuses on one of those key factors, the degree to which the teacher provides scaffolding that enables students to grapple with the task at hand without sacrificing or diluting the important mathematics in it.

The question is: Do students have opportunities to engage in *productive struggle*, through which they develop understandings of mathematics and the kind of perseverance in problem solving that is valued (CCSSM, 2010; NCTM, 1989, 2000; NRC, 2001)?

The challenge in instruction is to find the right balance. If students are spoon fed mathematics in bite-sized pieces, or told how to solve problems whenever they run into difficulties, they do not have the opportunity to build deep understandings and productive habits of mind. If they are not given scaffolds when they are not sure what to do, they are also deprived of opportunities to learn. The right level of scaffolding helps students to understand the challenges they confront, but leaves them room to make their own progress on those challenges.

A sharp contrast in levels of cognitive demand can be seen in the TIMSS videotapes (NCES, 1998) of American and Japanese geometry classes. In the U.S. classroom, the content (measures of complementary and supplementary angles) is *proceduralized* and the interactions between teacher and students take place primarily by way of IRE sequences (Mehan, 1979: the teacher Initiates the sequence by asking a typically short, factual question; the student Responds; and the teacher Evaluates the response, after which he or she moves on to another such IRE sequence). This form of interaction reduces mathematics to memorization and makes no demands for complex thinking on the part of the students. In the Japanese lesson, students were given the problem and ample time to make progress on it, after which the teacher called on various groups of students to present their ideas and explain the mathematics involved in their solution attempts. The students held the floor for some time in those exchanges. But, the students were not simply on their own. The teacher provided support to make sure the tasks were accessible to the students, and he pulled the presentations together – without depriving the students of the productive struggles that took place when they grappled with difficult problems.

The literature indicates that when students experience difficulty dealing with complex mathematical issues, there is a tendency for teachers to reduce cognitive demand, and thus to deprive students of the opportunities for productive struggle and sense making (see, e.g., Henningsen and Stein, 1997). The challenge for instruction is to provide clarifications and other support (e.g., heuristic advice, asking students if they have thought about particular issues or approaches), without telling students precisely what to do and thus denying them the opportunity to engage productively with mathematical challenges.

Ways Cognitive Demand can be productively supported in the mathematics classroom

There are many ways that teachers can initiate cognitively demanding activities in the classroom.

- Teachers, in the choice and design of tasks, can avoid providing detailed step-by-step instructions for solving problems, or repetitive exercises that mostly require students to apply memorized procedures. Rather, the materials/tasks chosen should be conceptually rich and provide ample opportunities for students to think and develop problem-solving skills.
- Teachers can actively support students in individual work, group work, and whole class discussions by asking clarifying questions and providing scaffolds, instead of moving directly to demonstrating solutions or suggesting specific methods to obtain solutions. (Examples of scaffolding questions can be found in all of the Formative Assessment Lessons developed by the Mathematics Assessment project, which are available at <<http://map.mathshell.org/materials/lessons.php>>.)
- Teachers can encourage students' productive struggle in a general way by discussing ideas of malleable intelligence and a growth mindset (Dweck, 2007), making it clear that learning mathematics is not a matter of memorization, and that one gets better at mathematics by working hard at it.

Caveats and Connections

This dimension focuses on the different levels of cognitive demand supported by the tasks the students are given, and the scaffolding the teacher provides. As the focus is on activities during the class period, the scheme may not reflect work done by the teacher in preparing the lesson, e.g., in selecting materials that offer students the appropriate amount of challenge. Aspects of task design and selection overlap in obvious ways with Dimension 1, The Mathematics. Rote exercises offer little opportunity for mathematical richness or sense-making, and little opportunity for productive struggle, thus they are likely to receive low scores in both dimensions. Potentially rich tasks could, however, be handled in the classroom in ways that the scores on dimensions 1 and 2 differ – e.g., if there are rich mathematical connections discussed by the teacher, but, the teacher scaffolds away difficulty when students encounter it. The key idea is that this dimension scores the extent to which students, given the materials at hand and supported by the teacher, have opportunities to grapple with challenging mathematical ideas in meaningful ways.

How TRU Math captures Cognitive Demand

The general rubric for assessing Cognitive Demand reads:

Cognitive Demand	
1	Classroom activities are structured so that students mostly apply memorized procedures and/or work routine exercises.
2	Classroom activities offer possibilities of conceptual richness or problem solving challenge, but teaching interactions tend to “scaffold away” the challenges, removing opportunities for productive struggle.
3	The teacher’s hints or scaffolds support students in productive struggle in building understandings and engaging in mathematical practices.

If the materials and tasks used during a particular classroom episode consist essentially of routine work sheets with detailed step-by-step procedures, or sets of repetitive exercises, then a score of 1 will be assigned. If the materials and tasks provide opportunities for the students to do some thinking and problem solving, but the teacher removes most of the challenge by channeling the student work in narrow directions, then a score of 2 will be assigned. If the materials provide room for thinking and problem solving, and the teacher’s comments provide scaffolding but still leave significant work for the students to do, then a score of 3 will be assigned.

Dimension 3: Access to Mathematical Content

The extent to which classroom activity structures invite and support the active engagement of all of the students in the classroom with the core mathematics being addressed by the class. No matter how rich the mathematics being discussed, a classroom in which a small number of students get most of the “air time” is not equitable.

Background

As noted in the overview to this document, an equitable and mathematically productive classroom provides *all* students access to meaningful mathematical content and practices – mathematics that is focused and coherent in a context that provides opportunities for students to develop their own understandings and build productive mathematical identities.

This dimension, *Access to Mathematical Content*, focuses on the question of whether there is uniform or differential access to the mathematics being addressed in the classroom. There may be mathematically rich discussions or other mathematically productive activities in the classroom – but, who gets to participate in them? If a subset of students is excluded from those conversations or activities, then they are being deprived of opportunities to learn. As discussed below, differential access may be a function of how the mathematics is framed or introduced, as well as who is called on to participate.

There is a long history of differential achievement in mathematics by students from varied racial, ethnic, and economic backgrounds (Secada, 1992), which, it has been argued, can be tied to differential access to opportunities to learn (Oakes, Joseph, & Muir, 2001). While one obvious source of this differential access is tracking, which is outside of the scope of a classroom observation scheme, another is the pattern of discourse within classrooms. Who has the opportunity to engage with mathematics in ways that are likely to lead to learning? Do all students have opportunities to discuss mathematical ideas with some frequency? In *How Schools Shortchange Girls* (American Association of University Women, 1992), for example, research revealed a pattern of boys being called upon more frequently than girls; and when they were called upon, girls were often asked questions that were less conceptually oriented than the questions that were asked of boys. Do multiple opportunities exist for students to engage with the mathematics, to develop and display competence (Cohen 1994), and to build understanding based on the knowledge they bring with them into the classroom (González et al., 2001; Zevenbergen, 2000)?

In brief, research indicates that effective teachers encourage participation by all students in the intellectual community of the classroom (Boaler, 2008; Cohen & Lotan, 1997; Schoenfeld, 2002). They select and set up tasks with a range of entry points to enable all students to engage in challenging mathematics. They set and reinforce expectations for various ways to participate in and contribute to classroom activities.

Access as it can play out in the mathematics classroom

There are numerous ways in which students can be deprived of, or supported in, access to the mathematics being addressed in the classroom.

- Teachers, both in choice and design of tasks, and in launching tasks, can provide multiple access points to the relevant mathematics, supporting the expectation that *all students* are able and expected to participate. (Tasks that can be done two or more ways, and in which solution methods can be related to each other, provide access to students who choose different pathways into the problem and, in addition, provide opportunities for making mathematical connections between student approaches.)
- Teachers can establish and reinforce norms of participation that encourage the generation and refinement of ideas rather than mainly critiquing or ignoring comments that are only partially correct.
- During mathematical discussions, teachers can encourage broad participation in a general way. For example, during whole class discussions or as they engage with groups, they can choose to call only on students who have not yet spoken. They can also use random methods to select students to contribute their ideas. These actions reinforce the notion that all students are able and expected to engage meaningfully in mathematical discussions.
- When students engage in problems that involve complex language and/or mathematical contexts, teachers can hold discussions to insure that all students have a grasp of the tasks and contexts at hand.
- As teachers engage in exposition of mathematical ideas, they can highlight multiple ways to think about and understand particular mathematical content, thus broadening potential access for students. For example, discussing multiple ways to represent a mathematical concept can make the concept accessible for students who are differentially comfortable with particular representations (as well as helping to make connections across representations, an aspect of focus, coherence, and sense-making).

Caveats and Connections

The emphasis of this dimension, as coded, is about *differential access* to the core mathematics being addressed by the class. There is much more to access writ large, which is not coded here.

Many of the factors that influence access to mathematics learning fall outside of the scope addressed by the rubrics included for dimension 3 of TRU Math. For instance, the level of cognitive demand inherent in the mathematics tasks made available to students, a central concern of dimension 2, is certainly an issue of access (Stein, Smith, Henningsen and Silver, 2000). All students may be deprived of the opportunity for productive struggle, in which case this issue will show up in TRU Math as a low score in dimension 2. Similarly, a classroom in which students have little opportunity to speak, except in direct response to teacher questions, will deprive all students of the opportunity to develop a mathematical voice (Boaler, 2008). This aspect of agency, authority, and identity will be coded in

Dimension 4. This dimension attends to the extent of *differential access among students* in the classroom being observed. That is, who is being considered, accommodated, engaged, and supported to engage in meaningful participation in the intellectual community of the classroom?

It is important to note, also, that some important elements of access to mathematics learning are outside of the scope of what can reliably be captured in discrete classroom observations. For example, many teachers work for long periods of time to create classroom norms that support equitable student participation in group work. While such norm setting is critical for building classroom culture that contributes to equitable access to learning, it is not likely to be observable during single, randomly selected lessons. The presence or absence of such norms may be capturable, however.

How TRU Math captures Access to Mathematical Content

Broadly speaking, *access to mathematical content* will be coded as a 1, 2, or 3 in TRU Math as follows:

Access to Mathematical Content	
1	There is differential access to or participation in the mathematical content, and no apparent efforts to address this issue.
2	There is uneven access or participation but the teacher makes some efforts to provide mathematical access to a wide range of students.
3	The teacher actively supports and to some degree achieves broad and meaningful mathematical participation; OR what appear to be established participation structures result in such engagement.

Consider a classroom in which there is powerful mathematics being discussed – between the teacher and just a few students who get called on repeatedly, while the rest of the classroom is left out of the conversation. Similarly, consider classrooms in which the “entry level” of the mathematics is sufficiently high that students who have some gaps in their backgrounds find themselves unsupported in trying to make sense of the problems they are supposed to work; or that they lack contextual knowledge that would help them mathematize a contextual problem. In such classrooms, even if the mathematics discussed can be high-powered, many students are disenfranchised from the mathematical sense making taking place. Such classrooms will be assigned scores of 1 on Dimension 3.

There are various ways to address such issues. If the teacher spends time discussing the context, and making sure that students have a sense of what is in it, there is a better chance students will be able to represent its features mathematically. If the teacher chooses problems that have multiple points of entry, so students can gain handholds into it, or encourages sharing of ideas both by the whole class and in group work, or if the teacher uses randomizing devices to call on students and then builds on what the students say, then the lesson will be assigned either a 2 or a 3. A score of 3 is reserved for observations where there is evidence of significant success of the efforts to provide universal access – hands go up across the class when the teacher asks a question, students across the class (or in small groups) are seen sharing ideas, and a significant cross-section of the class has and takes the

opportunity to do and talk about the mathematics. Evidence for a 3 may include the teacher actively encouraging participation, with success; or it may be the result of established classroom norms.

Again, Dimension 3 codes *equitable access to the core mathematics being addressed by the class*. The nature of that mathematics – e.g., how focused and coherent it is; how much cognitive demand it affords; and whether or not the class as a whole has opportunities for agency and authority – is coded in dimensions 1, 2, and 4 respectively.

Additional indicators of the distribution of student access to productive learning opportunities, not coded in Dimension 3, are the portion of class time spent on mathematical thinking, the amount of time that students are able to discuss their ideas with other students and the portion of students who contribute meaningfully to mathematical discussions. Students are more likely to learn important mathematics in classes that spend all or most of the observable time focusing on the mathematics they are learning, rather than administrative details or non-mathematical discussions. Also, more students are likely to have opportunities to engage in meaningful mathematical discussion in classes in which students are provided opportunities to discuss mathematics together in groups or in pairs. The process of breaking classroom sessions into episodes for coding, and recording the character of those episodes, will provide data on the degree to which students are given such opportunities.

Dimension 4: Agency, Authority, and Identity

The extent to which students have opportunities to conjecture, explain, make mathematical arguments, and build on one another's ideas, in ways that contribute to their development of agency (the capacity and willingness to engage mathematically) and authority (recognition for being mathematically solid), resulting in positive identities as doers of mathematics.

Dimension 4 attempts to capture the extent to which students have the opportunity to generate and share mathematical ideas, either publicly or in small groups; the extent to which authorship is recognized and supported; and the extent to which student ideas are built upon as the classroom constructs its collective mathematical understandings.

Background

Mathematical dispositions and identities – individuals' sense of who they are mathematically – are derived from experiences with mathematics, and they shape the ways in which these individuals go about doing mathematics. Many students develop counterproductive beliefs about themselves and mathematics, e.g., that they are “bad at math,” or that mathematics is to be memorized rather than understood, or that only geniuses can create mathematics (Schoenfeld, 1985). But it need not be this way. If students experience mathematics in ways that enable them to see themselves as mathematically productive or powerful, they can develop a sense of themselves as mathematical thinkers.

Two fundamental aspects of mathematical identity are agency and authority. In this context, *agency* refers to an individual's perception that she or he is someone who can do mathematics – that he or she can make progress on challenging mathematical problems by working away at them, and trust in the mathematical conclusions that he or she draws. This sense of self has obvious consequences for the ways in which one engages with mathematics.

Authority is a complex construct, with multiple meanings. Here we mean an individual's sense that he or she is knowledgeable about certain aspects of mathematics, and that he or she is recognized by others as having such knowledge.

In discussing these issues, Engle (2011) argues that “there are four kinds of intellectual authority that build upon each other: intellectual agency, authorship, contributorship, and finally being positioned as a local authority:”

“The first crucial, but most minimal, level of authority is having intellectual agency (Cobb et al., 1997; Lampert, 1990a, 1990b; Stipek et al., 1998). Learners have intellectual agency when they are “authorized” to share what they actually think about the problem in focus rather than feeling the need to come up with a response that they may or may not believe in, but that matches what some other authority like a teacher or textbook would say is correct (e.g., Hutchinson & Hammer, 2010; Lamon et al., 1996; Lehrer, Carpenter, Schauble, & Putz, 2000; Magnusson & Palincsar, 1995; Scardamalia et al., 1994). Learners' authority is then strengthened if they become publicly recognized as authors of their own ideas, what I now refer to as authorship (Lampert, 1990a,

1990b; O'Connor & Michaels, 1996; Toma 1991; Wertsch & Toma, 1995). The next form of strengthening is contributorship, or when a learner's authoring influences the ideas of others in his or her learning environment and beyond (Schwartz, 1999; cf. Engle, Langer-Osuna & McKinney de Royston, 2010). Finally, the strongest level of authority that a learner can have is to become socially recognized as an authority about some topic(s), which occurs gradually as his or her ideas become increasingly influential with others (Brown et al., 1993; Engle, Langer-Osuna & McKinney de Royston, 2010; Gresalfi, Martin, Hand & Greeno, 2009)."

The question, then, is to what extent a learning environment provides opportunities for students to develop these aspects of their mathematical selves.

Agency, Authority, and Identity as they can play out in the mathematics classroom

Research suggests that effective teachers recognize and capitalize on the strengths of each student, finding ways to help individual students enter into the learning community when they do not easily enter on their own (Boaler, 2008; Horn, 2007; Cohen & Lotan, 1997). Teachers accomplish this through actions that publicly recognize and reinforce the strengths and abilities of individual students, with the combined effect that they are each clearly expected and invited to be part of the classroom mathematical community. This includes attributing competence to students in class discussion and in group work, re-engaging students who are struggling, and challenging individual students to elaborate on their own ideas and the ideas of their classmates. Teachers can create opportunities for public recognition of students' contributions to mathematical discussions, help students work together in small groups, and attend to students who are struggling by building on the strengths in their thinking.

As teachers engage in the exposition of mathematical ideas, they can make connections to individual student ideas (e.g., by revoicing or naming students' contributions), thereby making clear that particular students have important ideas. In doing so, teachers assign academic status to individual students, increasing their access to the intellectual community. Resnick, O'Connor, and Michaels (2007) identify six important talk moves by teachers, describing them as follows:

- (1) Revoicing: "So let me see if I've got your thinking right. You're saying _____?" (with time for students to accept or reject the teacher's formulation);
- (2) Asking students to restate someone else's reasoning: "Can you repeat what he just said in your own words?";
- (3) Asking students to apply their own reasoning to someone else's reasoning: "Do you agree or disagree and why?";
- (4) Prompting students for further participation: "Would someone like to add on?";
- (5) Asking students to explicate their reasoning: "Why do you think that?" or "How did you arrive at that answer?" or "Say more about that";
- (6) Challenge or Counter Example: "Is this always true?" or "Can you think of any examples that would not work?"

During mathematical discussions and small group work, teachers can make efforts to involve particular students who might not otherwise be involved. During whole class discussions or as they engage with groups they can highlight particular student strengths, for example, and make explicit how those strengths are relevant and useful for particular mathematical purposes. (Such encouragement is most likely to become apparent when coding small group work, individual work, or student presentations).

Caveats and Connections

The original title of this dimension was “Agency, Authority, and Accountability” – the final “A” representing accountability to the discipline. It should be the case that as students are developing a sense of themselves as doers of mathematics, that the mathematics that they produce is focused, coherent, and, of course, correct. That is, classroom dialogue should be accountable to the discipline. “Accountable talk” refers to a form of classroom dialogue, described as follows by the Institute for Research on Learning:

For classroom talk to promote learning it must be accountable to appropriate knowledge, and to rigorous thinking. Accountable Talk seriously responds to and further develops what others in the group have said. It puts forth and demands knowledge that is accurate and relevant to the issue under discussion. Accountable Talk uses evidence appropriate to the discipline (e.g., proofs in mathematics, data from investigations in science, textual details in literature, documentary sources in history) and follows established norms of good reasoning. Teachers should intentionally create the norms and skills of Accountable Talk in their classrooms. (IRL, 2011)

Disentangling the opportunities to speak about mathematics and the opportunities to talk mathematics *correctly* is a challenge. For the most part, Dimension 1 (the mathematics) is intended to capture the accountability to the discipline. Dimension 1 addresses the extent to which classroom conversations are mathematically accountable; Dimension 4 addresses the extent to which students have opportunities to develop a mathematical voice and a corresponding sense of self, or identity.

Note also that Dimension 4 does not capture *which* students have the opportunity to develop a voice. If there is differential access to the classroom practices that support agency and identity, this be captured in Dimension 3.

How TRU Math captures (opportunities for) Agency, Authority, and Identity

As noted above, Dimension 4 attempts to capture the extent to which students have the opportunity to generate and share mathematical ideas, either publicly or in small groups; the extent to which “authorship” is recognized and supported, and the extent to which student ideas are built upon as the classroom constructs its collective mathematical understandings.

Agency, Authority, and Identity will be coded as a 1, 2, or 3 in TRU Math as follows:

Agency, Authority, and Identity	
1	The teacher initiates conversations. Students' speech turns are short (one sentence or less), and constrained by what the teacher says or does.
2	Students have a chance to explain some of their thinking, but "the student proposes, the teacher disposes": in class discussions, student ideas are not explored or built upon.
3	Students explain their ideas and reasoning. The teacher may ascribe ownership for students' ideas in exposition, AND/OR students respond to and build on each other's ideas.

A score of 1 for an episode means that the discussion is teacher-directed and focused; other than when the teacher responds to direct questions from students, the mathematics "comes from" the teacher. A score of 2 means that there is room for student explanation – but, under teacher control, and student ideas are not built upon. A score of 3 indicates that students' mathematical ideas are built upon through questioning, critique, connection, comparison, and/or extension.

Dimension 5: Uses of Assessment

The extent to which the teacher solicits student thinking and subsequent instruction responds to those ideas, by building on productive beginnings or addressing emerging misunderstandings. Powerful instruction “meets students where they are” and gives them opportunities to move forward.

Background

In contrast to commonplace classroom practices that position assessment as separate from instruction, serving a predominately evaluative function (Shepard, 2000), major policy documents in mathematics education clearly assert that assessment should be an integral component of instruction (NRC, 2005, 2001; NCTM, 1995). Black and Wiliam’s (1998) widely cited review of research literature documents the substantial learning gains that result from teachers’ use of formative assessment practices. When assessment becomes an integral and ongoing part of the learning process, as opposed to an interruption of classroom activities, students’ thinking takes on a more central role in determining the direction and shape of classroom activities (Shepard, 2000; Shafer & Romberg, 1999; de Lange, 1999). Assessment functions formatively when student reasoning is elicited and then meaningfully incorporated into a lesson. This happens when student reasoning is explicitly foregrounded as an object of inquiry and becomes a central object of discussion that guides the teacher in more adeptly supporting and enhancing students’ individual and collective reasoning (Webb & Romberg, 1992).

Teachers’ uses of assessment

In order for assessment to function formatively, information about student reasoning and understanding must not only be elicited, but the information gathered must be consequential in shaping classroom activities (Black et al, 2003). When eliciting information, teachers must craft tasks and ask purposefully formulated questions with the intent of obtaining information that will improve the process of learning (Shepard, 2000). This occurs when students’ sense making (e.g. strategies employed, types of formal and informal reasoning, depth of conceptual understanding, strategic use of procedures, connections made, problem solving, etc.) is a focal point of classroom activity. Crucially, in obtaining insight into the meaning students are making, a teacher creates the opportunity to refine instruction in response to students’ ways of knowing (NRC, 2001, p.349). Subsequent questions, tasks, and other “next-step” decisions can be made to direct learning towards important mathematical goals. Through this process of deliberately *hearing* student reasoning and understanding, and then shaping instruction in response, teaching “becomes clearer, more focused, and more effective” (NRC, 2001, p.350).

Positive ramifications resulting from student thinking being elicited and refined

When assessment becomes a central feature of a teacher’s classroom practice, students are repeatedly asked to articulate their thinking (NRC, 2005). According to Romberg, Carpenter, and Kwako (2007), “the ability to articulate ideas is a benchmark of understanding” (p.16). More critically, when students are asked to articulate what they learn either through explaining or justifying their reasoning, or

through summarizing the critical ideas in a task, they must necessarily be reflective. Through reflection, students become more conscious of their own learning and can be lead to make connections between what they are learning and what they have already learned (Carpenter & Lehrer, 1999). Students can “take control of their own learning, consciously define learning goals, and monitor their progress in advancing them” when they develop this metacognitive stance towards their own learning (NRC, 2005, p.10). Teachers should aim to make students’ thinking transparent to both themselves and to their students. Norms of classroom discourse can be created that allow students to feel inclined to share and critique each other’s reasoning (Shepard, 2000)⁵. In this way, a classroom culture can be created in which explanation and justification are expected, and reflection on learning is promoted.

A caveat – limitations inherent to the observation scheme

The ways in which a teacher alters classroom activities in response to student ideas may often be unobservable, because an outside observer does not have access to the moment-to-moment decisions made by the teacher. Accordingly, TRU Math focuses on the ways in which the instructor makes explicit the fact that student reasoning is being used to guide the lesson, while acknowledging that there may still be uses of student reasoning that are not captured, because the teacher does make explicit the ways that student reasoning is being built upon.

How our scheme captures particular uses of assessment

Are tasks framed in ways that can reflect and/or reveal student thinking?

Here we are concerned with the extent to which task framing allows for the incorporation of student thinking in the lesson. Does a task provide students with opportunities to explain, or does it just ask them to find answers? Then, when students do give explanations, is it simply so that the teacher can assess them, or does the task structure and/or classroom norms allow for student explanations to become objects of discussion in themselves? For instance, a teacher could provide opportunities for student ideas to guide the lesson by creating spaces for discussion during the lesson, rather than only during summary.

While presenting mathematical ideas, does the teacher explicitly reference student thinking? Is student thinking built on?

In order to be effective, teacher exposition should be targeted at students’ current understandings. Here we are concerned with evidence that a teacher’s explanations are grounded in student ideas and common misconceptions, “meeting students where they are” and building on and productively shaping the understandings they reveal. While the design of the lesson can certainly support this process, the teacher will need to elicit information through the use of questioning and observing students working on a task. In order to provide backing for a claim that a teacher is using student reasoning, we look for specific references to student ideas, such as talking about “Jose’s idea” and leading the class to build upon it.

During a mathematical discussion, how does the teacher build upon and refine student thinking?

⁵ Of course, such norms contribute to the development of student agency and authority as well. Here, the fact that nascent ideas are open for discussion contributes productive direction of classroom conversations.

When student reasoning is opened up to the class for discussion, what does the teacher do with it? Does the teacher push individual students to go further in their thinking? Does the teacher go one step further and push students to reason with one another's reasoning? In this way, student reasoning becomes an object of discussion, and thus can become an object that is used to guide the lesson.

Connection to Other Dimensions

The ability of a teacher to use assessment effectively is contingent upon many other aspects of a lesson. In order to build upon student reasoning, such reasoning must first be elicited (Dimension 4: agency, authority, and identity), and it should be elicited in an equitable fashion (Dimension 3: access to mathematical content). Moreover, student reasoning is most meaningful when it is with respect to deep engagement with mathematical tasks (Dimension 2: cognitive demand). In addition, as noted above, the use of formative assessment – in particular, the airing of student ideas in ways that refines and builds on them – can be seen as contributing to coherent mathematics (dimension 1) and to the development of student voice and identity (dimension 4).

Broadly speaking, uses of assessment will be coded in TRU Math as follows:

Uses of Assessment	
1	Student reasoning is not actively surfaced or pursued. Teacher actions are limited to corrective feedback or encouragement.
2	The teacher refers to student thinking, perhaps even to common mistakes, but specific students' ideas are not built on (when potentially valuable) or used to address challenges (when problematic).
3	The teacher solicits student thinking and subsequent instruction responds to those ideas, by building on productive beginnings or addressing emerging misunderstandings.

The criteria for a score of 1 are clear. A score of 2 indicates that the teacher's comments are generic; there is no clear indication that the teacher is helping students to understand the challenges or possibilities of particular ideas that emerge from student comments. A score of 3 indicates that student thinking is elicited and referred to, and that there is evidence that what is revealed about student thinking sometimes affects the direction of the class.

Content Elaboration for Contextual Algebraic Tasks

The extent to which students are supported in dealing with complex modeling and applications problems, which typically call for understanding complex problem contexts (most frequently described in text), identifying relevant variables and the relationships between them, representing those variables and relationships symbolically, operating on the symbols, and interpreting the results.

The content elaboration dimension of TRU Math is intended to capture the extent to which instruction supports students in developing algebraic competencies for purposes of modeling and problem solving. Our emphasis is on preparation for working with Contextual Algebraic Tasks (CATs) – word problems describing situations that call for algebraic symbolization and analysis, and for which there is not a readily available template that reduces the solution of the problem to an algorithm. Solutions to the classic genres of word problems (rate problems, age problems, mixture problems, etc.) are often taught as routines. While having the skills to work such tasks is important, we are primarily interested in the broader representational aspects of algebraic sense making and problem solving that comprise a deeper understanding of the content.

Background

It stands to reason that the opportunities students receive to work through CATs in various classroom settings—whole group, small group, individually—will affect their ability to solve them individually. The developers of the Connected Mathematics Project curriculum wrote,

Evidence from cognitive sciences' research supports the theory that students can make sense of mathematics if the concepts and skills are embedded in a context or a problem. If time is spent exploring interesting mathematics situations, reflecting on solution methods, examining why the methods work, comparing methods, and relating methods to those used in previous situations, then students are likely to build *more robust understanding* of mathematical concepts and related procedures. (Lappan, Phillips, & Fey, 1997, p. 73, emphasis added).

Our analyses of classroom video recordings and the literature led us to identify aspects of classroom activity that seem likely to promote development of the five aspects of robust understanding which we call the *robustness criteria* (RCs) for contextual algebraic tasks. We organize these aspects of classroom activity according to their associated aspects of robust understanding as follows:

Content Elaboration: “Robustness Criteria” for Contextual Algebraic Tasks

RC 1	Reading and interpreting text, and understanding the contexts described in problem statements.
RC 2	Identifying salient quantities in a problem and articulating relationships between them
RC 3	Using algebraic representations of those relationships, including:
	a) Generating algebraic representations of relationships between quantities
	b) Interpreting and making connections between representations
RC 4	Executing Algebraic Procedures and Checking Solutions.
	a) Executing calculations and procedures with precision
	b) Checking plausibility of results
RC 5	Explaining and justifying reasoning
	a) Opportunities for Student Explanations
	b) Teacher Instruction about Explanations
	c) Student Explanations and Justifications

A rationale for each RC (based on Lepak, Wernet, & Floden, in preparation) is provided in the following sections. This dimension, observing for RCs, is scored differently than dimensions 1 through 5. Each of dimensions 1 through 5 is always present in a classroom; thus the question for scoring is, how rich are the activities along that dimension? In contrast, on any given day, a class may or may not be working on contextual problems; and if they are, classroom discussions may or may not include activities closely connected with learning to work with contextual algebraic tasks. Hence, the robustness criteria are noted *when they appear*. If the activity indicated by the rubrics below does not appear in a teaching episode, then the RC is not coded.

Broadly speaking, a score of 1 on the rubrics that follow indicates a class focused on procedures, “local” situations, and facts rather than global relationships between quantities. A score of 2 generally represents more focus on covariation and generalization, and a score of 3 represents a functional approach to algebra that emphasizes covariation, generalization, sense making, explaining and justifying reasoning. As Chazan (1996) has argued, algebra instruction that views functions as central provides many points of access for a wide range of students by emphasizing relations among variables through a range of representations – including “verbal descriptions, graphs, sketches of graphs, tables of values, and algebraic expressions” (p. 462).

RC 1: Reading and interpreting text, and understanding the contexts described in problem statements

RC1 concerns making sense of the problem context. It goes without saying that students will make little progress if they are unable to make sense of the context described in a problem statement. To give an example from the literature, Walkington and colleagues (2012) study of high school students solving algebra word problems found that when students struggled with the demands of verbal interpretation, their ability to solve problems was impeded.

Problem representation is widely considered to be central to the problem solving process. As Pólya (1945) stated: “First, you have to understand the problem.” The first of Mayer’s four cognitive phases in problem solving is *problem representation*, in which the solver “constructs a mental representation of the situation described in the problem” (in Brenner et al., 1997, p. 665). Similarly, the first standard for mathematical practice identified in the Common Core State Standards -- Mathematics (CCSS-M) (Common Core State Standards Initiative, 2010) is making sense of problems; that is, “mathematically proficient students start by explaining to themselves the meaning of a problem and looking for entry points to its solution” (p. 6). Schoenfeld (2004) identified reading the problem, parsing the words correctly, and inferring the mathematical character of the solution from the wording of the problem as aspects of student knowledge needed to solve word problems. All of these aspects are involved in interpreting the text to make sense of the problem.

In the classroom, teachers can help students construct mental representations of the context by unpacking it through questioning and elaboration. This process assists students in obtaining an overall idea of the problem by defining terms, both mathematical and non-mathematical, clearing up ambiguities surrounding the language of the problem context, and anticipating feasible solutions. However, it is also important that students are learning how to do this work for themselves so that they will not always rely on the teacher to scaffold the sense-making process.

RC1 is coded as follows. Recall that RCs are coded only when they appear.

RC 1: Reading and interpreting text, and understanding the contexts described in problem statements

1	One or more terms in a task are reworded and/or defined, or a specific algebra-related term (e.g., function) in the text or lesson notes is defined
2	The context (problem scenario) is elaborated or discussed and an explicit attempt is made to ensure students understand it.
3	The teacher or students link the context (problem scenario) with algebraic concepts (e.g. rate of change, proportion, variable, expression).

RC 2: Identifying salient quantities in a problem and articulating relationships between them

Identifying relevant quantities is essential for students to successfully solve contextual algebraic tasks. Students need to identify which quantities are relevant to the situation (and desired solution) and how these quantities are related (Schoenfeld, 2004). This includes articulating the relationships between quantities, whether as a qualitative relationship, identification of a family of functions, or a generalization of pattern. These skills are particularly important because they serve as the foundation for representing the relationships between quantities. In order to represent a contextual task algebraically (RC3), students must first understand the relationships and how change in one variable affects change in the other(s).

One objective in the Algebra 1 Content Standards (NCTM, 2000) is for students to “identify essential quantitative relationships in a situation and determine the class or classes of functions that might model the relationship” (p. 665). One conceptualization of algebra itself is as the study of relationships among quantities (Usiskin, 1988). Similarly, Driscoll (1999) identified building rules to represent functions—considering how variables are changing and how the input is related to output by well-defined rules—as an algebraic habit of mind. Walkington and colleagues (2012) found that students were largely unsuccessful in solving algebra word problems when they jumped directly from the problem text to representation without making sense of quantities and the relationships between them.

“Defining appropriate quantities for the purpose of descriptive modeling” is a standard set by the CCSS-M (Common Core State Standards Initiative, 2010, N-Q 2). Implicit in this standard is that students first identify what quantities are relevant to the descriptive model. This requires students to sift through the given context to determine what information is relevant to solving the problem, and what information is extraneous. Once relevant quantities are identified, students need to determine how they are related.

A classroom activity that has promise for helping students develop these skills, is when teachers assign tasks requiring students to sort through the context for relevant quantities. Teachers can guide students toward determining relevant quantities through questions like, “What are we trying to find here?”, “What are we looking for?”, and “What do we know?”, and by calling attention to what quantities are changing and are needed to solve the problem.

Students also need to identify the relationships between relevant quantities in a Contextual Algebraic Task, and be able to articulate their reasoning about the relationships between those quantities. “Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects” (Common Core State Standards Initiative, p. 6). They may be helped to do so through identifying the family of functions relating quantities (e.g., “it’s linear”) or through a more qualitative characterization of the relationships

(e.g., “the perimeter goes up by 4 each time”). As implied with these examples, articulating a relationship requires students to verbally state the relationship rather than (and typically as a precursor to) using a symbolic representation.

In the classroom, the teacher can ask students to attend to how quantities are related by considering how one variable changes with another or by highlighting the features of a certain type of function. When working on a contextual task, the relationship between quantities should connect to the problem context. The teacher should support students in using multiplicative reasoning and generalization of relationships, as appropriate, in addition to additive or recursive reasoning.

RC2 is coded as follows. Recall that RCs are coded only when they appear.

RC 2: Identifying salient quantities in a problem and articulating relationships between them

1	Salient quantities are identified but the relationships between quantities are not discussed (e.g., “What are the slope and y-intercept?” “We know Jose’s speed, we need to find the distance he travelled.”)
2	Salient quantities are identified and local relationships between quantities are discussed (e.g. at a particular point: “What is the cost of plan A for 10 hours? Of Plan B?”)
3	General covariation of quantities is discussed (e.g. “As time increases, distance stays the same”; “When x increases by 1, y increases by 2”) or the relevant family of functions is identified.

RC 3: Using algebraic representations of relationships

Relating and representing multiple varying quantities is a core feature of a functions-based approach to algebra, which puts functional relationships among quantities, rather than abstract manipulation of symbols, at the central of high school algebra (Kieran, 2007; Kilpatrick & Izsak, 2008). We define algebraic representations to include coordinate graphs, bivariate tables, variable equations, and diagrams that link the context to salient quantities (Common Core State Standards Initiative, 2010; Cooney et al., 2010; Driscoll, 1999; NCTM, 2000). Generating and interpreting these representations is the core idea in multiple NCTM Algebra Standards. For example, the Understanding Patterns, Relations, and Functions Standard states that students should be able to “represent, analyze, and generalize a variety of patterns and tables, graphs, words, and when possible, symbolic rules” (NCTM, 2000, p.222). One of the CCSS-M mathematical practices (2010) is modeling with mathematics, mapping relationships between quantities with “diagrams, two-way tables, graphs, flowcharts, and formulas” (p. 7). We have split RC 3 into two components.

RC 3a: Generating algebraic representations of relationships between quantities

We consider generating a representation as either producing a representation from scratch or adding to an existing representation. Schoenfeld (2004) identified building a situation model and building a diagram or other appropriate representations as important aspects of student knowledge for solving word problems. Use of multiple representations facilitates students’ development of mathematical concepts (e.g., Brenner et al., 1997; Stein et al., 2008) and their efforts to carry out problem solving tasks (e.g., Greeno & Hall, 1997). Teachers may promote this ability by asking students to construct specific representations, as well as by giving students opportunities to choose a representation to construct.

RC 3b: Interpreting and making connections between representations

Interpreting a representation means drawing information from that representation (possibly to solve a problem); making connections involves using one representation to generate another, comparing the usefulness of multiple representations, or identifying how parameters in an equation appear in other representations. Using and moving between algebraic representations is emphasized in the literature on algebra learning as it leads to a broader and more integrated understanding of functions (Driscoll, 1999; Kieran, 2007).

In the classroom, teachers may help students interpret and make connections between representations through questioning that targets relationships between representations and how changes in one representation will affect others. For example, the teacher may ask students to make connections among representations, compare advantages or limitations of various representations, or focus on the parameters in an equation and how these parameters affect the features of other representations.

RC3 is coded as follows. Recall that RCs are coded only when they appear.

RC 3a: Generating representations of relationships between quantities

1	Algebraic representation(s) is(are) generated by way of practice (e.g., writing equation for a line given two points) without attention to the relationship(s) between variables or why the representation is a good choice for the given situation.
2	Algebraic representation(s) is(are) purposefully generated with explicit attention to either the relationship between variables <i>or</i> why the representation is a good choice for the given situation.
3	Algebraic representation(s) is(are) purposefully generated with explicit attention to the relationship between variables <i>and</i> attention to why the representation is a good choice for the given situation (e.g., "let's make a graph so we can see all the possible solutions to the equation").

RC 3b: Interpreting and making connections between representations

1	Representations are interpreted locally or in part (e.g., relevant quantities identified but relationship between quantities is not exploited, such as "From the graph, when x is 4, y is what?") There are no connections between multiple representations.
2	Important global features of representations are explicated to highlight the covariation between quantities (e.g., relating the 'steepness' of a graph to a rate of change, using the representation to identify the family of functions relating the quantities) <i>or</i> connections among multiple representations are explored (e.g. focus on parameters in an equation and how they affect the features of the representations, affordances of different representations may be highlighted).
3	Important global features of representations are explicated to highlight covariation between quantities <i>and</i> connections among multiple representations are explored (e.g. focus on parameters in an equation and how the parameters affect the features of the representations); affordances of different representations may be highlighted.

RC 4: Executing algebraic procedures and checking solutions

Solving complex algebraic problems typically involves one or more procedures or calculations to arrive at a solution. An important component of a robust understanding for working with contextual problems is that students monitor their solution strategies as well as checking their solution. When working with contextual problems students need to be able to solve the relevant equations as appropriate, and check to make sure the solution makes sense (Schoenfeld, 2004). As reflected in the NCTM Problem Solving Standard (2000), students need to consult the problem context to determine if their strategy needs to be adapted and if the solution is feasible within the confines of the context. We have separated RC #4 into two components, executing calculations and procedures with precision and checking plausibility of the results.

RC 4a: Executing calculations and procedures with precision

Students should be able to accurately execute algebraic and arithmetic procedures and calculations. This coincides with the strand of procedural fluency (NRC, 2001) which includes both knowing *how* to accurately perform procedures and knowing *when* to apply them. These concepts are reflected in multiple CCSS-M practices, most clearly *attend to precision* (Practice 6). Drawing on content standard documents and related literature (Common Core State Standards Initiative, 2010, NCTM, 2000), we have defined algebraic procedures as including, but not being limited to:

- Substituting a value or values into a variable expression and evaluating
- Solving linear equations and inequalities for a single variable
- Solving a proportion
- Solving a system of linear equations or inequalities through linear combinations or substitution
- Iterating recursive functions
- Finding equivalent expressions by distributing, combining like terms, etc.
- Performing arithmetic with polynomial and rational expressions
- Solving quadratic equations by factoring, completing the square, or applying the quadratic formula.

RC 4b: Checking plausibility of results

Students should be able to refer to the problem context and mathematical procedures to check the plausibility of their results and make sense of their solution. Indeed, part of procedural fluency includes anticipating the results of a computation and making sense of the results (NRC, 2001). Making sense of procedures and calculations and executing them with meaning also requires students to *look for and make use of structure* (Practice 7) through which they “step back for an overview and shift perspective” (p. 8), and *look for and express regularity in repeated reasoning*, through which they “maintain oversight of the process, while attending to details” (p. 8). All of this can be seen as related

to Pólya's (1945) "looking back" phase – now that a computational answer has been obtained, does that answer actually make sense in the context of the problem?

RC4 is coded as follows. Recall that RCs are coded only when they appear.

RC 4a: Executing calculations and procedures with precision

1	Arithmetic calculations are executed accurately, and any errors are corrected.
2	Algebraic procedures (similar to those listed above) are executed accurately, and any errors are corrected.
3	Calculations and/or algebraic procedures are executed correctly with explicit attention to accuracy, or mistakes are caught and instruction involves guiding students to self assess and correct their calculational/procedural errors.

RC 4b: Checking plausibility of results

1	The plausibility of a solution is passively checked (e.g. teacher poses the question, "Does this answer make sense?")
2	The plausibility of a solution is actively checked without attending to context (e.g., checking that the answer makes sense with regard to a representation or calculation, but not with the context).
3	The plausibility of a solution is actively checked in relationship to the context (problem scenario) to make sense of the solution (i.e. to judge the meaning, utility, and reasonableness of the results; NCTM, 2000, p. 296)

RC 5: Explaining and justifying reasoning

An algebraic explanation or justification involves clarification of students' algebraic reasoning (Yackel, 2001) – that is, providing support for their answer using algebraic reasons. Being able to explain and justify one's work is important in all mathematics courses. Graham, Cuoco, and Zimmerman (2010) argue that these are important mathematical skills beginning from kindergarten. These skills are also included in the NCTM Reasoning and Proof Standard (2000), which calls for students to develop and evaluate mathematical arguments and proofs and select and use various types of reasoning and methods of proof. More recently, making a mathematical argument is featured as a mathematical practice in CCSS-M, *making a viable argument and critiquing the reasoning of others*.

A promising way to support development of students' capacity for explaining and justifying reasoning is for teachers to provide and support opportunities for students to explain their thinking and justify why it makes sense algebraically. In doing so, teachers will go beyond requiring students to provide only an answer, to asking students to provide justification for why their statements make mathematical sense. This may be achieved through talk moves such as teacher questioning and revoicing, but may also be extended to asking students to question and revoice each others' ideas. (Note that such explanations are related to Dimension 1, The Mathematics, and Dimension 4, Agency, Authority, and Identity.)

The TRU Math rubric codes for instances of students developing their abilities to explain and justify their reasoning through opportunities in class to do so (RC5a) and instances of teacher support for building understanding about what qualifies as mathematically adequate justifications (RC5b). Finally, the rubric records the quality of the student explanations themselves (RC5c).

RC5 is coded as follows. Recall that RCs are coded only when they appear.

RC 5a: Opportunities for Student Explanations

1	An open-ended question is posed for students without explicitly soliciting an explanation or justification.
2	An explanation is explicitly requested of students, but the nature of the explanation is not specific or does not require an algebraic justification (e.g. "Why?", "an you explain that?")
3	An explanation is explicitly requested that focuses on algebraic reasoning (e.g. an algebraic representation, the qualitative relationship between quantities, or the problem context).

RC 5b: Teacher Instruction about Explanations

1	Teacher explicitly provides guidelines on what is needed <i>generally</i> for good explanations
2	Teacher explicitly provides guidelines on what is <i>generally</i> needed for good explanations <i>and</i> models such behavior.
3	Teacher provides feedback on and/or opportunities for students to incorporate the feedback to revise <i>specific explanations</i> .

RC 5c: Student Explanations and Justifications

1	Student gives a short explanation that describes only procedures (whether algebraic or non-algebraic), OR the explanation is unclear.
2	Student describes procedures, supporting them by either referring to the problem context or the underlying mathematical concepts.
3	Student generates a clear algebraic explanation (e.g. draws on an algebraic representation, the qualitative relationship between quantities, or the problem context) that extends beyond explaining how to do a procedure.

Summary

As a whole, the rubric dimension for “Robustness Criteria” for Contextual Algebraic Tasks is meant to capture classroom activities that have promise for helping students develop the robust understanding that will allow them to succeed in making sense of and working with contextual algebraic tasks. That robust understanding does not suggest that successfully working with such tasks should be a linear sequence of steps (i.e. Step 1: Interpret language, Step 2: Identify quantities, and so on). Rather, a robust understanding for working with contextual algebraic task is an integrated set of knowledge and skill that can be drawn on when interpreting and mathematizing algebraic situations in and out of the classroom.

Positive Ramifications Resulting from Teaching for Robust Understanding

In addition to the benefits listed above, instruction that targets the development of a robust understanding of algebra has promise for moving students beyond using formulas and algorithms to effectively develop algebraic habits of mind (Driscoll, 1999). That is, teachers may model and support

students in sense-making, modeling, solving using a variety of techniques, and making sense of the solution. With the development of these skills, students are able to take a multi-faceted and flexible approach to working with contextual problems. The goal is to have student become “doers of mathematics” (Silver, 1994), whose habits of mind support their engaging in mathematical sense making.

Limitations of the content elaboration in TRU Math

As with any observational rubric, selecting what to code for in this content elaboration involved difficult choices regarding what we would and would not capture. One such choice was whether to capture the actors in classroom events relevant to the RCs—the teacher or students. That is, who was taking an active role in developing the algebraic ideas? In the end, we consciously decided to make the rubrics in the content elaboration actorless, believing that: a) regardless of who introduced and developed RC-specific ideas, all students in the classroom would benefit from the access to those ideas, and b) the level of student access and agency would be captured elsewhere in the rubric.

Another limitation of the content elaboration dimension rubrics is the need to focus on specific aspects of RCs. For example, in the RC 3 rubrics we did not distinguish between generating something more traditionally algebraic like an equation, and a table or diagram that shows relationships between quantities (i.e., some might “rank” generating an equation above, say, a table, and score it higher.) Instead, we capture the focus of the representations—whether the focus is on covariation and generalization, or local values for individual quantities.

Connections to Other Dimensions

This content elaboration is focused on the knowledge related to solving contextual algebraic tasks, and as such can be seen as zooming in on a particularly important aspect of algebra learning. Classroom discussions of CATS, of necessity, involve dimensions 1 through 5. Dimension 1 characterizes the coherence and sense making of the discussion; one would expect that high scores on the content elaboration for contextual algebraic tasks would be associated with high scores on dimension 1. Opportunities to explain and justify (RC5) are clearly related to cognitive demand (dimension 2) and the development of agency, authority, and identity (dimension 4).

How TRU Math Captures the Development of Robust Algebraic Understanding

This content elaboration for contextual algebraic tasks captures the type of instruction we believe has promise for promoting students’ *robust understanding of algebra* in the sense described by Lappan, Phillips, & Fey (1997) at the beginning of this section. In its focus on representation and interpretation, this dimension, unlike many other observation protocols, captures instruction targeted towards a functions-based approach to algebraic instruction.

This content elaboration contains nine algebra-specific rubrics addressing aspects of robust understanding of algebra. The two parts of RC 2 have been grouped together for classroom coding, and RC 5 maps to three different rubrics (one to capture opportunities for students to explain their

reasoning, one to capture the teacher asking for explanation, and one to capture provision of guidelines on what counts as a good explanation). Scoring considerations are as follows.

Recall that classroom activities related to the robustness criteria may or may not appear in any given lesson – the lesson may or may not deal with word problems or contextual algebraic tasks, and even if it does, for example, students may or may not have an opportunity to explain their reasoning or make connections between representations. Hence this content elaboration dimension is only coded when the teacher or students are explicitly engaged in the RC-related activities describe above—that is, language is interpreted, quantities and their relationships are highlighted, procedures and calculations are executed, and so on. Then, scores from 1 to 3 are assigned to capture the richness of the opportunity for students to develop competencies allowing for success with CATs. As noted above, scores of 1 on these rubrics tend to indicate a class focused on procedures, “local” situations, and facts, rather than global relationships between quantities. Scores of 2 generally represent more focus on covariation and generalization, and scores of 3 represents a functional approach to algebra that emphasizes covariation, generalization and sense making.

A concluding note on the relationship between the content elaboration and the mathematical practices in the Common Core State Standards for Mathematics:

As one might expect, a classroom that supports the robust understanding of algebra must support students in the development of the mathematical practices identified in the CCSS-M. A rough mapping between the robustness criteria in the content elaboration and the mathematical practices (MP) in the CCSS-M is as follows:

Robustness Criteria		Mathematical Practice
RC1	Supports & Is Supported By	MP1, MP2, & MP4.
RC2	Supports & Is Supported By	MP1, MP2, MP4, MP7, & MP8
RC3	Supports & Is Supported By	MP2, MP4, MP7, MP8, & MP5
RC4	Supports & Is Supported By	MP2, MP6 & possibly MP5
RC5	Supports & Is Supported By	MP3 and MP6

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